THE ESSENTIAL SPECTRUM OF THE LAPLACIAN

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Abstract. In this article we prove a generalization of Weyl's criterion for the essential spectrum of a self-adjoint operator on a Hilbert space. We then apply this criterion to the Laplacian on functions over open manifolds to generalize the set of manifolds for which the L^2 essential spectrum is $[0,\infty)$. We prove that the L^2 essential spectrum is $[0,\infty)$ whenever the volume of the manifold does not decay exponentially and its Ricci curvature in the radial direction is asymptotically nonnegative. In fact we prove an even more general result, namely that the L^2 essential spectrum is $[0,\infty)$ whenever the Laplacian of the radial function at infinity is asymptotically nonnegative in the sense of distribution and the volume of the manifold does not decay exponentially. We also use our criterion to compute the essential spectrum of a complete shrinking Ricci soliton and of manifolds that posses an exhaustion function. Finally we add some remarks on the relationship between the L^p spectrum of the manifold and volume growth.

1. Introduction

Let M be a complete noncompact Riemannian manifold of dimension n and denote by Δ the Dirichlet Laplacian. It is well known that $-\Delta$ is a nonnegative definite and densely defined self-adjoint operator on $L^2(M)$. The essential spectrum of the Laplacian on functions has been extensively studied. It is known that on hyperbolic space a spectral gap appears and was initially conjectured that when a manifold has Ricci curvature nonnegative then its essential spectrum is the nonnegative real line $[0,\infty)$. It was however impossible to prove this conjecture by directly computing the L^2 spectrum without any further assumptions on the curvature and geometry of the manifold (see for example [4, 8, 9, 11, 12, 15, 23]). The theorem of K. T. Sturm [21] that

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the essential spectrum of the Laplacian Δ_q on $L^q(M)$ is independent of q whenever the volume of the manifold grows uniformly subexponentially and its Ricci curvature is bounded below, was seminal in generalizing the above results. Using this theorem, Wang (and later Lu-Zhou) was able to generalize the conjecture to manifolds with Ricci curvature asymptotically nonnegative [22], [20]. This assumption on curvature implies that the volume of the manifold grows uniformly subexponentially (cf. [3, 21]). The key idea was to compute the L^1 essential spectrum instead, which is in general a much simpler task, to prove that the L^1 spectrum is $[0,\infty)$. He then used Sturm's result to conclude that the L^2 spectrum must also be $[0,\infty)$.

Our goal is to generalize the set of manifolds for which the L^2 spectrum is the nonnegative real line. The idea that this can only happen whenever the L^q spectra are the same, seemed too strong of a condition to impose and, as we will see, is not necessary. In this article we will circumvent Sturm's Theorem, and prove instead a generalization of Weyl's criterion for the essential spectrum. As a result, we will no longer need to assume uniformly subexponential volume growth for the manifold. Instead, we will only suppose that Ricci curvature is asymptotically nonnegative in the radial direction and in the case that the volume of the manifold is finite, we will make the further assumption that its volume does not decay exponentially. Our condition on curvature only imposes subexponential volume growth at a point. We will prove that on such manifolds the L^2 essential spectrum is $[0,\infty)$.

Our first main application of the generalization of Weyl's criterion will be

Theorem 1.1. *Let M be a complete noncompact Riemannian manifold. Suppose that, with respect to a fixed point p, the radial Ricci curvature is asymptotically nonnegative (see Lemma 4.2), and if the volume of the manifold is finite we additionally assume that its volume does not decay exponentially at p. Then the L^2 spectrum of the Laplace operator on functions is* $[0,\infty)$.

In fact we will be able to prove a more general, albeit more technical result

Theorem 1.2. *Let M be a complete noncompact Riemannian manifold. Suppose that, with respect to a fixed point p, the radial function $r(x) = d(x,p)$ satisfies*

$$\overline{\lim_{r\to\infty}}\,\Delta r \le 0$$

in the sense of distribution, and if the volume of the manifold is finite, we additionally assume that its volume does not decay exponentially at p. Then the L^2 spectrum of the Laplace operator on functions is $[0,\infty)$.

The above theorems generalize significantly the results of Wang [22] and Lu-Zhou [20], since both articles assumed Ricci curvature asymptotical nonnegative and their conditions implied uniformly subexponential volume growth.

In § 5.1, we provide two different proofs for Theorems 1.1 and 1.2, based on two generealized Weyl's criteria that we develop for the Laplacian on functions (see Corollaries 5.2, 5.3). While with the former we use cut-off functions when building our test functions, as is the usual practice in the literature, with the latter we only need appropriate continuous functions.

We will also see that in the case of a manifold with a pole, or in a warped product form, the above theorems reduce to simpler assumptions on the manifold (see Proposition 5.1 and Remark 5.1).

Moreover, we will use the generalization of Weyl's criterion to compute the L^2 essential spectrum of complete shrinking Ricci solitons with no assumptions on curvature. In our last application, we will show how it can be used to modify a result of Elworthy-Wang [10] on manifolds that posses an exhaustion function.

We end our article with some remarks on the volume growth of the manifold. We demonstrate that the L^q independence of the spectrum has only been proved so far for the case of uniformly subexponential volume growth.

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2. The Weyl Criterion for Quadratic Forms

Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. The norm and inner product in $\mathcal{H}$ are respectively denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Let $\sigma(H), \sigma_{\text{ess}}(H)$ be the spectrum and the essential spectrum of H, respectively. The Classical Weyl criterion states that

Theorem 2.1 (Classical Weyl's criterion)**.** *A nonnegative real number λ belongs to $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ such that*

(1) $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1$,

(2) $(H-\lambda)\psi_n \to 0$, *as* $n \to \infty$ *in* $\mathcal{H}$.

Moreover, λ belongs to $\sigma_{\text{ess}}(H)$ of H if, and only if, in addition to the above properties

(3) $\psi_n \to 0$ *weakly as* $n \to \infty$ *in* $\mathcal{H}$.

Remark 2.1. *The above theorem is still true if the convergence in (2) is replaced by weak convergence, the statement of which can be found (without proof) in* [7] *and later in* [19]*. This version of the Weyl criterion was applied for the first time to the Laplacian on curved Euclidean domains in* [19]*. The authors are grateful to David Krejčiřík for informing them of the results which led to the following generalization.*

Theorem 2.2. *Let f be a bounded positive continuous function over $[0,\infty)$. A non-negative real number λ belongs to the spectrum $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ such that*

(1) $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1$,

(2) $(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) \to 0,$ *as* $n \to \infty$ *and*

(3) $(\psi_n, (H-\lambda)\psi_n) \to 0,$ *as* $n \to \infty$.

Moreover, λ belongs to $\sigma_{\rm ess}(H)$ of H if, and only if, in addition to the above properties

(4) $\psi_n \to 0,$ *weakly as* $n \to \infty$ *in* $\mathcal{H}$.

Proof. Since H is a densely defined self-adjoint operator, the spectral measure E exists and we can write

$$H = \int_0^\infty \lambda \, dE. \tag{1}$$

Assume that $\lambda \in \sigma(H)$. Then by Weyl's criterion, there exists a sequence $\{\psi_n\}$ such that

$$\|(H-\lambda)\psi_n\| \to 0, \quad \|\psi_n\| = 1$$

as $n \to \infty$.
We write

$$\psi_n = \int_0^\infty dE(t)\psi_n$$

as its spectral decomposition. Then

$$(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) = \int_0^\infty f(t)(t-\lambda)^2 d\|E(t)\psi_n\|^2.$$

Since f is a bounded positive function, we have

$$(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) \le C \int_0^\infty (t-\lambda)^2 d\|E(t)\psi_n\|^2 = C\|(H-\lambda)\psi_n\|^2.$$

Similarly,

$$(\psi_n, (H-\lambda)\psi_n) \le C \, \|\psi_n\| \cdot \|(H-\lambda)\psi_n\|.$$

Thus the necessary part of the theorem is proved.

Now assume that $\lambda > 0$ and $\lambda \notin \sigma(H)$. Then there is a $\lambda > \varepsilon > 0$ such that $E(\lambda+\varepsilon) - E(\lambda-\varepsilon) = 0$. We write

$$\psi_n = \psi_n^1 + \psi_n^2,$$

where

$$\psi_n^1 = \int_0^{\lambda-\varepsilon} dE(t)\psi_n,$$

and $\psi_n^2 = \psi_n - \psi_n^1$.
Then

$$\begin{aligned}
&(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) \\
&= (f(H)(H-\lambda)\psi_n^1, (H-\lambda)\psi_n^1) + (f(H)(H-\lambda)\psi_n^2, (H-\lambda)\psi_n^2) \\
&\geq c_1\|\psi_n^1\|^2 + (f(H)(H-\lambda)\psi_n^2, (H-\lambda)\psi_n^2) \geq c_1\|\psi_n^1\|^2,
\end{aligned}$$

where the positive number c_1 is the infimum of the function $f(t)(t-\lambda)^2$ on $[0, \lambda-\varepsilon]$. Therefore

$$\|\psi_n^1\| \to 0$$

by *(2)*. On the other hand, we similarly get

$$(\psi_n, (H-\lambda)\psi_n) \geq \varepsilon\|\psi_n^2\|^2 - \lambda\|\psi_n^1\|^2.$$

If the criteria *(2)*, *(3)* are satisfied, then, by the two estimates above, we conclude that both ψ_n^1, ψ_n^2 go to zero. This contradicts $\|\psi_n\| = 1$, and the theorem is proved. Note that for $\lambda = 0$, ψ_n^1 is automatically zero, and the second half of the proof would give the contradiction.

□

Remark 2.2. *We note that the above result does not require the construction of the approximate eigenfunctions in the computation of the spectrum of the operator. This is in contrast to Weyl's criterion. Nevertheless, it provides an alternative way to compute the spectrum which will prove to be simpler in certain cases of interest.*

In this article we will apply Theorem 2.2 to the Laplacian on functions. In this setting two particular cases of the function f will be of interest.

Corollary 2.1. *A nonnegative real number λ belongs to the spectrum $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ such that*

(1) $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1$,
(2) $((H+1)^{-1}\psi_n, (H-\lambda)\psi_n) \to 0$, *as* $n \to \infty$ *and*

(3) $(\psi_n, (H-\lambda)\psi_n) \to 0$, *as* $n \to \infty$.

Moreover, λ *belongs to* $\sigma_{\mathrm{ess}}(H)$ *of* H *if, and only if, in addition to the above properties*

(4) $\psi_n \to 0$, *weakly as* $n \to \infty$ *in* $\mathcal{H}$.

Proof. We take $f(x) = (x+1)^{-1}$. The corollary follows from the identity

$$(H+1)^{-1}(H-\lambda) = 1 - (\lambda+1)(H+1)^{-1}.$$

□

In a similar spirit, taking $f(x) = (x+\alpha)^{-(N+1)}$ for a natural number N and a positive number $\alpha > 1$, we also obtain the following generalization

Corollary 2.2. *A nonnegative real number* λ *belongs to the spectrum* $\sigma(H)$ *if, and only if, there exists a sequence* $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ *such that*

(1) $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1$,
(2) $((H+\alpha)^{-i}\psi_n, (H-\lambda)\psi_n) \to 0$ *as* $n \to \infty$ *for two consecutive natural numbers i=N, N+1, and*
(3) $(\psi_n, (H-\lambda)\psi_n) \to 0$, *as* $n \to \infty$.

Moreover, λ *belongs to* $\sigma_{\mathrm{ess}}(H)$ *of* H *if, and only if, in addition to the above properties*

(4) $\psi_n \to 0$, *weakly as* $n \to \infty$ *in* $\mathcal{H}$.

3. An Approximation Theorem

Let M be a complete noncompact Riemannian manifold. Let $p \in M$ be a fixed point. Define

$$r(x) = d(x,p)$$

to be the radial function on M. It is well known that

(1) $r(x)$ is continuous;
(2) $|\nabla r(x)| = 1$ almost everywhere and $r(x)$ is a Lipschitz function;
(3) Δr exists on $M\backslash\{p\}$ in the sense of distribution.

In general, it is not possible to find smooth approximations of a Lipschitz function under the $\mathcal{C}^1$ norm. The following Proposition, which is a more precise version of [20, Proposition 1], implies that this can be done up to a function with small L^1 norm. Such kind of result is essential in Riemannian geometry and should be well-known, but given that we were not able to find a reference, we include a proof.

Proposition 3.1. *For any positive continuous decreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\lim_{r\to\infty} \eta(r) = 0,$$

there exist C^∞ functions $\tilde{r}(x)$ and $b(x)$ on M such that

(a). $\|b\|_{L^1(M\setminus B_p(R))} \le \eta(R-1)$;

(b). $\|\nabla\tilde{r} - \nabla r\|_{L^1(M\setminus B_p(R))} \le \eta(R)$

and for any $x \in M$ with $r(x) > 2$

(c). $|\tilde{r}(x) - r(x)| \le \eta(r(x))$ and $|\nabla\tilde{r}(x)| \le 2$;

(d). $\Delta\tilde{r}(x) \le \max_{y\in B_x(1)} \Delta r(y) + \eta(r(x)) + |b(x)|$ in the sense of distribution.

Proof. Without loss of generality, we assume that $\eta(r) < 1$. Let $\{U_i\}$ be a locally finite cover of M and let $\{\psi_i\}$ be the partition of unity subordinate to the cover. Let $\mathbf{x_i} = (x_i^1, \cdots, x_i^n)$ be the local coordinates of U_i. Define $r_i = r|_{U_i}$.

Let $\xi(\mathbf{x})$ be a non-negative smooth function on $\mathbb{R}^n$ whose support is within the unit ball. Assume that

$$\int_{\mathbb{R}^n} \xi = 1.$$

Without loss of generality, we assume that each U_i is an open subset of the unit ball of $\mathbb{R}^n$ with coordinates $\mathbf{x_i}$. Then for any $\varepsilon > 0$,

$$r_{i,\varepsilon} = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \xi\left(\frac{\mathbf{x_i} - \mathbf{y_i}}{\varepsilon}\right) r_i(\mathbf{y_i}) d\mathbf{y_i}$$

is a smooth function on U_i and hence on M. Let $\{\sigma_i\}$ be a sequence of positive numbers such that

$$\sum_i \sigma_i(|\Delta\psi_i(x)| + 4|\nabla\psi_i(x)| + \psi_i(x)) \le \eta(r(x)). \tag{2}$$

By [13, Lemma 7.1, 7.2], for each i, we can choose $\varepsilon_i < 1$ small enough so that

$$\begin{aligned} &|r_{i,\varepsilon_i}(x) - r_i(x)| \le \sigma_i; \\ &\|\nabla r_{i,\varepsilon_i} - \nabla r_i\|_{L^1(U_i)} \le \sigma_i. \end{aligned} \tag{3}$$

We also have

$$\Delta r_{i,\varepsilon_i}(x) \le \max_{y\in B_x(1)} \Delta r_i(y). \tag{4}$$

Define

$$\tilde{r} = \sum_i \psi_i r_{i,\varepsilon_i}, \quad b = 2\sum_i \nabla\psi_i \cdot \nabla r_{i,\varepsilon_i}.$$

Since $\sum_i(\nabla\psi_i\cdot\nabla r_i)=(\sum_i\nabla\psi_i)\cdot\nabla r=0$ almost everywhere on M, we have

$$b=2\sum_i\nabla\psi_i\cdot(\nabla r_{i,\varepsilon_i}-\nabla r_i)$$

almost everywhere. Thus *(a)* follows. Similarly, observing that

$$\tilde r-r=\sum_i\psi_i(r_{i,\varepsilon_i}-r_i),\quad\text{and}\quad|\nabla r_{i,\varepsilon_i}|<2,$$

we obtain *(b)*, *(c)*.

To prove *(d)*, we compute

$$\Delta\tilde r=\sum_i[(\Delta\psi_i)\,r_{i,\varepsilon_i}+2\nabla\psi_i\nabla r_{i,\varepsilon_i}+\psi_i\Delta r_{i,\varepsilon_i}],$$

and since

$$\sum_i(\Delta\psi_i)r_i=\sum_i(\Delta\psi_i)r=0,$$

we have

$$\Delta\tilde r=\sum_i[\Delta\psi_i(r_{i,\varepsilon_i}-r_i)+b+\psi_i\Delta r_{i,\varepsilon_i}].$$

By (4), we obtain *(d)* and the Proposition is proved.

□

4. Manifolds with Δr Asymptotically Nonpositive

As we have mentioned in the previous section, the Laplacian of the radial function $r(x)=d(x,p)$ exists in the sense of distribution (except at p). That is, for any nonnegative smooth function f with compact support in $M\backslash\{p\}$, the integral

$$\int_M f\Delta r$$

is defined. The following simple observation is due to Wang [22] and is crucial in our estimates.

Lemma 4.1. *The function Δr is locally integrable away from p.*

Proof. Let W be any compact set of the form $B_p(R)-B_p(r)$ for $R>r>0$. Then by the Laplacian comparison theorem, there is a constant C, depending only on the dimension, r, R, and the lower bound of the Ricci curvature on $B_p(R)$, such that

$$\Delta r\le C$$

on W in the sense of distribution. Thus we have

$$|\Delta r| = |C - \Delta r - C| \leq 2C - \Delta r$$

and therefore

$$\int_W |\Delta r| \leq 2C \operatorname{vol}(W) - \int_W \Delta r.$$

Using Stokes' Theorem, we obtain

$$\int_W |\Delta r| \leq 2C \operatorname{vol}(W) - \int_{\partial W} \frac{\partial r}{\partial n} \leq 2C \operatorname{vol}(W) + \operatorname{vol}(\partial W),$$

and the lemma is proved.

□

In general the Laplacian of the radial function is not locally L^2, which makes the direct computation of the L^2 essential spectrum difficult.

Take $M = S^1 \times (-\infty, \infty)$ for example, letting (θ, x) be the coordinates. Then the radial function with respect to the point $(0, 0)$ is given by

$$r = \sqrt{x^2 + (\min(\theta, 2\pi - \theta))^2}.$$

A straightforward computation gives

$$\Delta r = -\frac{2\pi}{\sqrt{x^2 + \pi^2}} \delta_{\{\theta = \pi\}} + \text{ a smooth function,}$$

where $\delta_{\{\theta=\pi\}}$ is the Delta function along the submanifold $\{\theta = \pi\}$. Therefore Δr is not locally L^2 (but it is locally L^1, by Lemma 4.1).

In this section, we study manifolds with the following property

$$\overline{\lim_{r \to \infty}} \Delta r \leq 0 \tag{5}$$

in the sense of distribution, where $r(x)$ is the radial distance of x to a fixed point p. We shall give a precise estimate of the L^1 norm of Δr in terms of the volume growth of the manifold. But before we do that, we first provide an important example where the above technical condition holds.

We note that for a fixed point $p \in M$ the cut locus $\operatorname{Cut}(p)$ is a set of measure zero in M. The manifold can be written as the disjoint union $M = \Omega \cup \operatorname{Cut}(p)$, where Ω is star-shaped with respect to p. That is, if $x \in \Omega$, then the geodesic line segment $\overline{px} \subset \Omega$. $\partial r = \partial / \partial r$ is well defined on Ω. We have the following result

Lemma 4.2. *Let $r(x)$ be the radial function with respect to p. Suppose that there exists a continuous function $\delta(r)$ on $\mathbb{R}^+$ such that*

(i). $\lim_{r\to\infty} \delta(r) = 0$
(ii). $\delta(r) > 0$ *and*
(iii). $\mathrm{Ric}(\partial r, \partial r) \geq -(n-1)\delta(r)$ *on* Ω.

Then (5) *is valid in the sense of distribution.*

Proof. On Ω, we have the following Bochner formula

$$0 = \frac{1}{2}\Delta|\nabla r|^2 = |\nabla^2 r|^2 + \nabla r \cdot \nabla(\Delta r) + \mathrm{Ric}(\partial r, \partial r).$$

Since $\nabla^2 r(\partial r, \partial r) = 0$, using the Cauchy inequality, we have

$$0 \geq \frac{1}{n-1}(\Delta r)^2 + \frac{\partial}{\partial r}(\Delta r) - (n-1)\delta(r). \tag{6}$$

Since Ω is star-shaped, for any fixed direction $\partial/\partial r$, we obtain (5) by comparing the above differential inequality with the Riccati equation.

On the points where r is not smooth, we may use the trick of Gromov as in Proposition 1.1 of [16] to conclude the result in the sense of distribution. □

4.1. **Volume comparison theorems.** Let p be the fixed point of the manifold. Denote

$$B(r) = B_p(r), \quad V(r) = \mathrm{vol}\,(B_p(r))$$

the geodesic ball of radius r at p and its volume respectively.

The following volume comparison theorem is well-known.

Lemma 4.3. *Let $r(x)$ be the radial function defined above. Assume that* (5) *is valid in the sense of distribution. Then the manifold has subexponential volume growth at p. In other words, for all $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$, depending only on ε and the manifold, such that for all $R > 0$*

$$V(R) \leq C(\varepsilon)\, e^{\varepsilon R}.$$

Proof. Let $m(r)$ be a nonnegative continuous function such that

$$\lim_{r\to\infty} m(r) = 0,$$

and

$$\Delta r \leq m(r)$$

in the sense of distribution. It follows that

$$\int_{B(R)\setminus B(1)} \Delta r \le \int_{B(R)\setminus B(1)} m(r)$$

which, by Stokes' Theorem, implies that

$$\mathrm{vol}\,(\partial B(R)) - \mathrm{vol}\,(\partial B(1)) \le \int_{B(R)\setminus B(1)} m(r).$$

Let $\varepsilon > 0$. Then we can find an R_ε such that $m(r) < \varepsilon$ for $r > R_\varepsilon$. Setting $f(R) = V(R)$, we obtain

$$f'(R) \le \mathrm{vol}\,(\partial B(1)) + \int_{B(R_\varepsilon)\setminus B(1)} m(r) + \varepsilon(f(R) - f(R_\varepsilon))$$

for any $R > R_\varepsilon$. Thus

$$(e^{-\varepsilon R}(f(R) - f(R_\varepsilon)))' \le C_\varepsilon e^{-\varepsilon R}$$

for $R > R_\varepsilon$, where C_ε is a constant depending on ε and the manifold M. Integrating from R_ε to R, we obtain

$$f(R) < f(R_\varepsilon) + C_\varepsilon \varepsilon^{-1} e^{-\varepsilon R_\varepsilon} e^{\varepsilon R}$$

for $R > R_\varepsilon$. Thus for any R, we have

$$V(R) = f(R) < C(\varepsilon) e^{\varepsilon R}$$

for

$$C(\varepsilon) = f(R_\varepsilon) + C_\varepsilon \varepsilon^{-1} e^{-\varepsilon R_\varepsilon}.$$

$\square$

In other words, whenever the Laplacian of the radial function $r(x) = d(x,p)$ is asymptotically nonnegative in the sense of distribution, the manifold has subexponential volume growth with respect to the point p. In the case of finite volume for the manifold M, we will also need an assumption on the decay rate of the volume of a ball of radius r. We say that the volume of M *decays exponentially at* p, if there exists an $\varepsilon_o > 0$ such that

$$\mathrm{vol}\,(M) - V(r) \le e^{-\varepsilon_o r}$$

for r large. For the purposes of computing the L^2 essential spectrum, we will need that the volume does not decay exponentially.

In the general case, our weak assumption on the decay of Δr does not provide any information on the decay rate of the volume. In dimension $n = 2$ if the manifold

satisfies $(i), (ii), (iii)$ as in Lemma 4.2, then its volume cannot decay exponentially. We believe that the same should hold true in higher dimensions, but the problem is still open. On a manifold with a pole at p that has finite volume, we can prove that the volume does not decay exponentially whenever the radial Ricci curvature is asymptotically nonnegative. The following volume comparison result seems to be new.

Proposition 4.1. *Suppose that M is a manifold with a pole at a point p that has finite volume and satisfies $(i), (ii), (iii)$ as in Lemma 4.2. Then the volume of M cannot decay exponentially.*

Proof. We let r be the radial function and $\theta \in S^{n-1}$. Let (r, θ) be the polar coordinate system and $J(r, \theta)$ be the volume element on the geodesic sphere.

Since $|\nabla r| = 1$, the Hessian, $\nabla^2 r$, of the function r is the second fundamental form on the geodesic sphere of radius r. By definition, $\mathrm{Tr}(\nabla^2 r) = \Delta r = H$ is the mean curvature on each geodesic sphere. Furthermore, by the formula for the Laplacian in local coordinates, we get

$$\Delta r = (n-1)\frac{J'}{J},$$

where the derivative is taken with respect to r.

Using inequality (6), we get

$$J''(r, \theta) \leq \delta(r) J(r, \theta). \tag{7}$$

We want to prove that for each $\varepsilon > 0$ there exists a $\tilde{C}(\varepsilon)$ such that

$$J(r, \theta) \geq \tilde{C}(\varepsilon) e^{-\varepsilon r} \tag{8}$$

for almost all $\theta \in S^{n-1}$, where the constant $\tilde{C}(\varepsilon)$ depends only on ε and the manifold M. To prove this, we fix θ and consider

$$h(r) = e^{-\varepsilon r} J'(r, \theta) + \varepsilon\, e^{-\varepsilon r} J(r, \theta).$$

Given the upper bound on J'' we have

$$h'(r) \leq 0$$

for $r \geq R_{\varepsilon^2}$. Since the manifold has finite volume, we can easily see that, in almost all θ directions, there exists a sequence $R_i \to \infty$ such that $J(R_i, \theta), J'(R_i, \theta) \to 0$.

Thus $h(r) \geq 0$ for $r > R_{\varepsilon^2}$, or $J'/J \geq -\varepsilon$. The latter implies (8). By continuity, the above lower bound for $J(r,\theta)$ is valid for any θ. Therefore

$$\mathrm{vol}\,(M) - V(r) = \int_r^\infty \int_{S^{n-1}} J^{n-1}(R,\theta)\,d\theta\,dR \geq (\tilde{C}(\varepsilon))^{n-1}\,e^{-\varepsilon(n-1)r}.$$

Since the ε is arbitrary, it follows that the volume of M does not decay exponentially. $\square$

We remark that if the volume of the manifold grows uniformly subexponentially, then its volume cannot decay exponentially at any point. The proof is simple, but we include it for the sake of completion. By the definition of uniformly subexponential volume growth, for all $\varepsilon > 0$ there exists a uniform constant $C(\varepsilon)$ such that

$$\mathrm{vol}\,(B_x(r)) \leq C(\varepsilon)\,\mathrm{vol}\,(B_x(1))\,e^{\varepsilon r}$$

for all $x \in M$. For $r > 1$ we let $y \in M$ be a point such that $d(x,y) = r-1$. By the uniformly subexponential volume growth of the manifold and since $B_x(1) \subset B_y(r)$, we have

$$\mathrm{vol}\,(B_y(1)) \geq (C(\varepsilon))^{-1}\,\mathrm{vol}\,(B_y(r))\,e^{-\varepsilon r} \geq (C(\varepsilon))^{-1}\mathrm{vol}\,(B_x(1))e^{-\varepsilon r}.$$

Therefore,

$$\mathrm{vol}\,(M) - \mathrm{vol}\,(B_x(r-1)) \geq \mathrm{vol}\,(B_y(1)) \geq (C(\varepsilon))^{-1}\mathrm{vol}\,(B_x(1))e^{-\varepsilon r},$$

and the volume of M can not decay exponentially.

4.2. L^1 **estimates for** $\Delta\tilde{r}$**.** We set $\tilde{r}$ to be the smoothing of r from Proposition 3.1. The following lemma is a more precise version of [20, Lemma 2].

Lemma 4.4. *Let $r(x)$ be the radial function to a fixed point p on M, and suppose that* (5) *is valid in the sense of distribution. Then we have the following two cases*

(a) Whenever $\mathrm{vol}\,(M)$ *is infinite, for any $\varepsilon > 0$ and $r_1 > 0$ large enough, there exists a $K = K(\varepsilon, r_1)$ such that for any $r_2 > K$, we have*

$$\int_{B(r_2)\setminus B(r_1)} |\Delta\tilde{r}| \leq \varepsilon\,V(r_2+1); \tag{9}$$

(b) Whenever $\mathrm{vol}\,(M)$ *is finite, for any $\varepsilon > 0$ there exists a $K(\varepsilon) > 0$ such that for any $r_2 > K$, we have*

$$\int_{M\setminus B(r_2)} |\Delta\tilde{r}| \leq \varepsilon\,(\mathrm{vol}\,(M) - V(r_2)) + 2\mathrm{vol}\,(\partial B(r_2)).$$

Proof. By Proposition 3.1 and using the idea in the proof of Lemma 4.1, we obtain

$$|\Delta\tilde{r}(x)| \leq 2(\max_{y\in B_x(1)} \Delta r(y) + \eta(r(x)) + |b(x)|) - \Delta\tilde{r}(x)$$

in the sense of distribution. Using our assumptions on Δr and η, we see that for any $\varepsilon > 0$ we can find an $r_1 > 0$ large enough such that whenever $r(x) > r_1$, then

$$2(\max_{y\in B_x(1)} \Delta r(y) + \eta(r(x))\,) < \varepsilon/2$$

also in the sense of distribution. Therefore for $r > r_1 + 2$,

$$\int_{B(r)\backslash B(r_1)} |\Delta\tilde{r}| \leq \frac{\varepsilon}{2}\,(V(r) - V(r_1)) + 2\int_{M\backslash B(r_1)} |b| - \int_{B(r)\backslash B(r_1)} \Delta\tilde{r}.$$

Using Stokes' Theorem, we get

$$\int_{B(r)\backslash B(r_1)} |\Delta\tilde{r}| \leq \frac{\varepsilon}{2}\,(V(r) - V(r_1)) + 2\int_{M\backslash B(r_1)} |b| - \int_{\partial B(r)} \frac{\partial\tilde{r}}{\partial n} + \int_{\partial B(r_1)} \frac{\partial\tilde{r}}{\partial n},$$

where $\partial/\partial n$ is the outward normal direction on the boundary. Obviously, the above implies that

$$\begin{aligned}\int_{B(r)\backslash B(r_1)} |\Delta\tilde{r}| \leq & \frac{\varepsilon}{2}\,(V(r) - V(r_1)) + 2\int_{M\backslash B(r_1)} |b| \\ & + \int_{\partial B(r)} \left|\frac{\partial\tilde{r}}{\partial n} - 1\right| + \int_{\partial B(r_1)} \frac{\partial\tilde{r}}{\partial n}.\end{aligned} \tag{10}$$

We first consider the case when the volume of M is infinite. By Proposition 3.1, choosing r_1 large enough we obtain

$$\int_{M\backslash B(r_1)} |b| < \frac{\varepsilon}{4}$$

and

$$\|\nabla\tilde{r} - \nabla r\|_{L^1(M\backslash B(r_1))} \leq 1 \tag{11}$$

Since the volume of M is infinite, then there exists $K = K(\varepsilon, r_1) > r_1 + 2$ such that whenever $r > K$

$$\int_{B(r)\backslash B(r_1)} |\Delta\tilde{r}| \leq \frac{3\varepsilon}{4}\,(V(r) - V(r_1)\,) + \int_{\partial B(r)} \left|\frac{\partial\tilde{r}}{\partial n} - 1\right|. \tag{12}$$

We choose an r' such that $|r'-r|<1$ and

$$\int_{\partial B(r')}\left|\frac{\partial\tilde r}{\partial n}-1\right|\leq\int_{r-1}^{r+1}\int_{\partial B(t)}\left|\frac{\partial\tilde r}{\partial n}-1\right|dt.$$

By (11), we have

$$\int_{\partial B(r')}\left|\frac{\partial\tilde r}{\partial n}-1\right|<2.$$

Therefore,

$$\int_{B(r')\setminus B(r_1)}|\Delta\tilde r|\leq\frac{3\varepsilon}{4}\left(V(r')-V(r_1)\right)+2.$$

Choosing a possibly larger $K(\varepsilon,r_1)$ we get *(a)*.

The proof of *(b)* is similar. We choose $\eta(r)$ decreasing to zero so fast so that

$$\int_{M\setminus B(r_1)}|b|\leq\frac{\varepsilon}{8}(\mathrm{vol}\,(M)-V(r_1)).$$

Since the volume of M is finite, sending $r\to\infty$ in (10) we have

$$\int_{M\setminus B_p(r_1)}|\Delta\tilde r|\leq\varepsilon\left(\mathrm{vol}\,(M)-V(r_1)\right)+\int_{\partial B_p(r_1)}\frac{\partial\tilde r}{\partial n}.$$

Since $|\partial\tilde r/\partial n|\leq 2$ by *(c)* of Proposition 3.1, the lemma follows.

$\square$

Corollary 4.1. *Suppose that $(i),(ii),(iii)$ hold on M as in Lemma 4.2. Then the same integral estimates for $\Delta\tilde r$ hold as in Lemma 4.4.*

5. The L^2 Spectrum.

5.1. **Construction of functions for Weyl's Criterion.** In this section,we let $\tilde r(x)$ be the smoothing function defined in Proposition 3.1 of the radial function $r(x)=d(x,p)$. For each $i\in\mathbb{N}$, let x_i,y_i,R_i,μ_i be large positive numbers such that $x_i>2R_i>2\mu_i+4$ and $y_i>x_i+2R_i$. We take the cut-off functions $\chi_i:\mathbb{R}^+\to\mathbb{R}^+$, smooth with support on $[x_i/R_i-1,y_i/R_i+1]$ and such that $\chi_i=1$ on $[x_i/R,y_i/R]$ and $|\chi_i'|,|\chi_i''|$ bounded. Let $\lambda>0$ be a positive number. We let

$$\phi_i(x)=\chi_i(\tilde r/R_i)\,e^{\sqrt{-1}\sqrt{\lambda}\,\tilde r}. \tag{13}$$

Setting $\phi = \phi_i$, $R = R_i$, $x = x_i$ and $\chi = \chi_i$, we compute

$$\Delta\phi + \lambda\phi = (R^{-2}\chi''(\tilde r/R) + 2i\sqrt{\lambda}R^{-1}\chi'(\tilde r/R))e^{\sqrt{-1}\sqrt{\lambda}\tilde r}|\nabla\tilde r|^2 \\ - \lambda\phi(|\nabla\tilde r|^2 - 1) + (R^{-1}\chi'(\tilde r/R) + i\sqrt{\lambda}\chi)e^{\sqrt{-1}\sqrt{\lambda}\tilde r}\Delta\tilde r.$$

Then we have

$$|\phi| \le 1, \quad |\Delta\phi + \lambda\phi| \le \frac{C}{R} + C|\Delta\tilde r| + C|\nabla\tilde r - \nabla r|, \tag{14}$$

where C is a constant depending only on λ and M.

Denote the inner product on $L^2(M)$ by $(\cdot\,,\,\cdot)$. We have the following key estimates

Lemma 5.1. *Suppose that* (5) *is valid for the radial function r in the sense of distribution. In the case that the volume of M is finite, we make the further assumption that its volume does not decay exponentially at p. Then there exist sequences of large numbers x_i, y_i, R_i, μ_i such that the supports of the ϕ_i are disjoint and*

$$\frac{\|(\Delta + \lambda)\phi_i\|_{L^1}}{(\phi_i, \phi_i)} \to 0$$

as $i \to \infty$.

Proof. The proof is similar to that of [20]. We define x_i, y_i, R_i, μ_i inductively. If $(x_{i-1}, y_{i-1}, R_{i-1}, \mu_{i-1})$ are defined, then we only need to let μ_i large enough so that the support of ϕ_i is disjoint with the previous ϕ_j's. For simplicity we suppress the i in our notation. The upper bound estimates for $|\phi|$ and $|\Delta\phi + \lambda\phi|$ given in (14)imply that

$$\begin{aligned}\int_M (\phi, \Delta\phi + \lambda\phi) \le & \frac{C}{R}\left[V(y+R) - V(x-R)\right] \\ & + C\int_{B(y+R)\setminus B(x-R)} |\Delta\tilde r| + \eta(x-R).\end{aligned} \tag{15}$$

When the volume of M is infinite, we choose a function η as in Proposition 3.1 such that $\eta \le 1$. By Lemma 4.4, if we choose R, x large enough but fixed, then for any $y > 0$ large enough we have

$$\int_M (\phi, \Delta\phi + \lambda\phi) \le 2\varepsilon\, V(y + R + 1).$$

Since $\|\phi\|_2^2 \ge V(y) - V(x)$, if we choose y large enough, $\|\phi\|_2^2 \ge \frac{1}{2}V(y)$. The subexponential volume growth of M at p that was proved in Lemma 4.3 implies that there

exists a sequence of $y_k \to \infty$ such that $V(y_k + R + 1) \le 2\,V(y_k)$. If not, then for a fixed number y and for all $k \in \mathbb{N}$ we have that

$$V(y + k(R+1)) > 2^k\,V(y).$$

However, by the subexponential volume growth of the manifold

$$2^k\,V(y) < V(y + k(R+1)) \le C(\varepsilon_1)\,e^{\varepsilon_1 y}\,e^{k\,\varepsilon_1(R+1)}$$

for any $\varepsilon_1 > 0$ and k large. This leads to a contradiction when we choose ε_1 such that $\varepsilon_1 R < \log 2$. Therefore, there exists a y such that

$$V(y + R + 1) \le 2\,V(y) \le 4\|\phi\|_2^2.$$

Combing the above inequalities, we have

$$\int_M (\phi, \Delta\phi + \lambda\phi) \le 8\varepsilon\|\phi\|_2^2.$$

We now consider the finite volume case. Using equation (15) and Lemma 4.4 we obtain for $x - R > K(\varepsilon)$

$$\begin{aligned}\int_M (\phi, \Delta\phi + \lambda\phi) \le& (R^{-1} + \varepsilon)\,[\mathrm{vol}\,(M) - V(x-R)] \\ &+ 2C\,\mathrm{vol}\,(\partial B(x-R)) + \eta(x-R).\end{aligned}$$

We set $h(r) = \mathrm{vol}\,(M) - V(r)$, a decreasing function. We choose $\eta(r)$ as in Proposition 3.1 so that $\eta(r) \le \frac{\varepsilon}{8}h(r)$. Making ε even smaller and choosing R and $x - R$ large enough, we get

$$\int_M (\phi, \Delta\phi + \lambda\phi) \le \varepsilon\,h(x-R) - 2C\,h'(x-R).$$

Given that $\|\phi\|_2^2 \ge h(x) - h(y)$ and the volume of M is finite, we can choose y large enough so that

$$\|\phi\|_2^2 \ge \frac{1}{2}h(x).$$

We would like to prove in this case that there exists a sequence of $x_k \to \infty$ such that

$$\varepsilon\,h(x_k - R) - 2C\,h'(x_k - R) \le 2\varepsilon h(x_k).$$

If the above inequality does not hold, then for all x large enough

$$\varepsilon\,h(x - R) - 2C\,h'(x - R) > 2\varepsilon h(x).$$

Replacing ε by $\varepsilon/2C$, we obtain

$$\varepsilon\, h(x-R) - h'(x-R) > 2\varepsilon h(x).$$

This implies that

$$-\big(e^{-\varepsilon x} h(x-R)\big)' > 2\varepsilon h(x)\, e^{-\varepsilon x}.$$

Integrating from x to $x+R$ and using the monotonicity of h we have

$$h(x-R) > 2(1-e^{-\varepsilon R}) h(x+R).$$

Choosing R even larger, we can make $2(1-e^{-\varepsilon R}) > 5/4$, therefore

$$h(x-R) > \frac{5}{4} h(x+R)$$

for all x large enough. By iterating this inequality, we get for all positive integers k

$$h(x-R) > \left(\frac{5}{4}\right)^k h(x+(2k-1)R).$$

Therefore

$$\mathrm{vol}\,(M) - V(x-R) > \left(\frac{5}{4}\right)^k [\mathrm{vol}\,(M) - V(x+(2k-1)R)\,]$$

which gives

$$\mathrm{vol}\,(M) - V(x+(2k-1)R) \le \left(\frac{4}{5}\right)^k \mathrm{vol}\,(M).$$

Sending $k\to\infty$ this contradicts the nonexponential decay assumption on the volume.

□

Corollary 4.1 gives

Corollary 5.1. *Suppose that $(i),(ii),(iii)$ hold on M as in Lemma 4.2. In the case that the volume of M is finite, we make the further assumption that its volume does not decay exponentially at p. Then there exist sequences of large numbers x_i, y_i, R_i, μ_i and cut-off functions χ_i such that the supports of the ϕ_i are disjoint and*

$$\frac{\|(\Delta+\lambda)\phi_i\|_{L^1}}{(\phi_i,\phi_i)} \to 0$$

as $i\to\infty$.

5.2. **A particular case of the generalized Weyl Criterion for the Laplacian.** In this subsection we will prove a special version of Theorem 2.2 for the Laplacian on functions. We begin with the fact that its resolvent is always bounded on L^∞.

Lemma 5.2. *We have*

$$(-\Delta+1)^{-1}$$

is bounded from $L^\infty(M)$ to itself.

The lemma follows from the proof of Lemma 3.1 in [2]. The resolvent is bounded on L^∞ because the heat kernel is bounded on L^∞. This is a property that Davies proves for any nonnegative self-adjoint operator that satisfies Kato's inequality like the Laplacian [5, Theorems 1.3.2,1.3.3]. It is due to the well-known fact that the Laplacian on functions is a self-adjoint operator that satisfies Kato's inequality. Together with Corollary 2.1 this lemma allows us to obtain an even simpler criterion for the essential spectrum of the Laplacian on functions:

Corollary 5.2. *A nonnegative real number λ belongs to the spectrum $\sigma(-\Delta)$, if there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of smooth functions such that*

(1) $$\frac{\|\psi_n\|_{L^\infty}\cdot\|(-\Delta-\lambda)\psi_n\|_{L^1}}{\|\psi_n\|_{L^2}^2}\to 0,\ \text{as } n\to\infty.$$

Moreover, λ belongs to $\sigma_{\mathrm{ess}}(-\Delta)$ of Δ, if

(2) For any compact subset K of M, there exists an n such that the support of ψ_n is outside K.

Proof. Without loss of generality, we may assume that $\|\psi_n\|_{L^2}=1$ for all n. By the above lemma, we have

$$|(\psi_n,(-\Delta-\lambda)\psi_n)|\le\|\psi_n\|_{L^\infty}\cdot\|(-\Delta-\lambda)\psi_n\|_{L^1}$$
$$|((-\Delta+1)^{-1}\psi_n,(-\Delta-\lambda)\psi_n)|\le C\|\psi_n\|_{L^\infty}\cdot\|(-\Delta-\lambda)\psi_n\|_{L^1}$$

for some constant C independent of n. Thus λ belongs to $\sigma(-\Delta)$ by Corollary 2.1. Moreover, if assumption *(2)* is valid, then λ belongs to the essential spectrum. $\square$

5.3. **The essential spectrum results.** We now combine the results of the previous sections to prove the main applications of Theorem 2.2.

Proof of Theorem 1.1 and Theorem 1.2. Let ϕ_i be the sequences of functions in (13), then by the construction of the functions, *(2)* of Corollary 5.2 is satisfied. Since $\|\phi_i\|_{L^\infty}=1$, *(1)* is valid by Lemma 5.1 and Corollary 5.1 in the respective cases. This completes the proof of the theorems. $\square$

In the case of a manifold with a pole at p the volume decay estimate of Proposition 4.1 allows us to obtain the following result, which is also new.

Proposition 5.1. *Let M be a complete noncompact Riemannian manifold with a pole at a point p. Suppose that, with respect to p, the radial Ricci curvature is asymptotically nonnegative in the sense of Lemma 4.2. Then the L^2 spectrum of the Laplace operator on functions is $[0,\infty)$.*

Remark 5.1. *We note that a similar result should hold on warped product manifolds $M = \mathbb{R} \times_J \tilde{M}$ with metric $g = d\rho^2 + J^2(\rho,\theta)\,\tilde{g}$, where $(\tilde{M},\tilde{g})$ is a compact $(n-1)$-dimensional submanifold of M and ρ is the distance function from this submanifold. Under the same asymptotically nonnegative assumption on $\mathrm{Ric}(\partial\rho,\partial\rho)$ as in Lemma 4.2, we also get that the L^2 spectrum of the Laplace operator on functions is $[0,\infty)$.*

5.4. **The use of continuous test functions.** In this subsection we will see that it is not necessary to use cut-off functions in our test functions. We will do that by first proving yet another version of the generalized Weyl's criterion (Corollary 5.3). This version of Weyl's Criterion sometimes provides a cleaner method for computing the essential spectrum.

Let D be a bounded domain of M with smooth boundary. We use the notation $\mathcal{C}_0^\infty(D)$ to denote the set of smooth functions on the closure $\bar{D}$ which vanish on the boundary ∂D. Let $\rho : D \to \mathbb{R}$ be the distance function to the boundary ∂D.

Definition 5.1. *We define $\mathcal{C}_0^+(D)$ to be the set of functions f on D with the properties*

1. *f is continuous, vanishing on ∂D;*
2. *f is Lipschitz, ∇f is essentially bounded, and $|\Delta f|$ exists in the sense of distribution;*
3. *As $\varepsilon \to 0$, $\int_{\{\rho\le\varepsilon\}} |f| \le \frac{1}{2}\varepsilon^2(\int_{\partial D}|\nabla f| + o(1))$, and $\int_{\{\rho\le\varepsilon\}}|\nabla f| \le \varepsilon(\int_{\partial D}|\nabla f| + o(1))$.*

Let $\mathcal{C}_0^+(M)$ be the set of continuous functions whose support is a bounded domain of M with smooth boundary and

$$f \in \mathcal{C}_0^+(\mathrm{supp} f).$$

We have the following

Corollary 5.3. *A nonnegative real number λ belongs to the spectrum $\sigma(-\Delta)$, if there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of functions in $\mathcal{C}_0^+(M)$ such that*

(1) $$\frac{\|\psi_n\|_{L^\infty(D_n)}\cdot(\|(-\Delta-\lambda)\psi_n\|_{L^1(D_n)}+\|\nabla\psi_n\|_{L^1(\partial D_n)})}{\|\psi_n\|^2_{L^2(D_n)}}\to 0,\ \textit{as } n\to\infty,$$

where $D_n = \operatorname{supp}\psi_n$. *Moreover,* λ *belongs to* $\sigma_{\mathrm{ess}}(-\Delta)$ *of* Δ, *if*

(2) For any compact subset K *of* M, *there exists an* n *such that the support of* ψ_n *is outside* K.

The above corollary can be proved using the following approximation result

Proposition 5.2. *Let* $f \in \mathcal{C}_0^+(M)$. *Then for any* $\varepsilon > 0$, *there exists a smooth function* h *of* M *such that*

(a) $\operatorname{supp}(h)\subset\operatorname{supp}(f)$;
(b) $\|f-h\|_{L^1}+\|f-h\|_{L^2}\le\varepsilon$;
(c) $\|(-\Delta-\lambda)h\|_{L^1}\le C(\|(-\Delta-\lambda)f\|_{L^1(D)}+\|\nabla f\|_{L^1(\partial D)})$,

where C *is a constant independent of* f, *and* $D=\operatorname{supp}(f)$.

Proof. Let $\chi(t)$ be a cut-off function such that it vanishes in a neighborhood of 0 and is 1 for $t\ge 1$. Let $\delta>0$ be a small number. Consider

$$g_\delta(x)=\chi\left(\frac{\rho(x)}{\delta}\right)f(x).$$

It is not difficult to prove *(a)*, *(b)* in the Proposition when we replace h by g_δ. To prove *(c)* we compute

$$(-\Delta-\lambda)g_\delta=\chi(-\Delta-\lambda)f-2\delta^{-1}\chi'\nabla\rho\nabla f-(\delta^{-2}\chi''+\delta^{-1}\chi'\Delta\rho)f.$$

Since ∂D is smooth, ρ is a smooth function near ∂D. Therefore by (3) of Definition 5.1 we have

$$\|(-\Delta-\lambda)g_\delta\|_{L^1}\le C(\|(-\Delta-\lambda)f\|_{L^1(D)}+\|\nabla f\|_{L^1(\partial D)})$$

for δ sufficiently small.

The proof that g_δ can be approximated by a smooth function is similar to that of Proposition 3.1. We sketch the proof here.

Let $D=\cup U_i$ be a finite cover of D. Without loss of generality, we assume that those U_i's which intersect with ∂D are outside the support of g_δ. Let $\mathbf{x_i}=(x_i^1,\cdots,x_i^n)$ be the local coordinates of U_i. Define $g_i=g_\delta|_{U_i}$.

Let $\xi(\mathbf{x})$ be a non-negative smooth function of $\mathbb{R}^n$ whose support is within the unit ball. Assume that

$$\int_{\mathbb{R}^n}\xi=1.$$

Without loss of generality, we assume that each U_i is an open subset of the unit ball of $\mathbb{R}^n$ with coordinates $\mathbf{x_i}$. Then for any $\varepsilon > 0$,

$$g_{i,\varepsilon} = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \xi \left(\frac{\mathbf{x_i} - \mathbf{y_i}}{\varepsilon} \right) g_i(\mathbf{y_i}) d\mathbf{y_i}$$

is a smooth function on U_i and hence on M. Let $\{\sigma_i\}$ be a sequence of positive numbers such that

$$\sum_i \sigma_i(|\Delta\psi_i(x)| + 4|\nabla\psi_i(x)| + \psi_i(x)) \tag{16}$$

is sufficiently small. By [13, Lemma 7.1, 7.2], for each i, we can choose $\varepsilon_i < 1$ small enough so that

$$\begin{aligned} &|g_{i,\varepsilon_i}(x) - g_i(x)| \le \sigma_i; \\ &\|\nabla g_{i,\varepsilon_i} - \nabla g_i\|_{L^1(U_i)} \le \sigma_i. \end{aligned} \tag{17}$$

We also have

$$\|\Delta g_{i,\varepsilon_i}\|_{L^1} \le \|\Delta g_i\|_{L^1}. \tag{18}$$

Define

$$h = \sum_i \psi_i g_{i,\varepsilon_i}, \quad b = 2\sum_i \nabla\psi_i \cdot \nabla g_{i,\varepsilon_i}.$$

Since $\sum_i (\nabla\psi_i \cdot \nabla g_i) = (\sum_i \nabla\psi_i) \cdot \nabla g_\delta = 0$ almost everywhere on D, we have

$$b = 2\sum_i \nabla\psi_i \cdot (\nabla g_{i,\varepsilon_i} - \nabla g_i).$$

We compute

$$\Delta h = \sum_i [(\Delta\psi_i)\, g_{i,\varepsilon_i} + 2\nabla\psi_i \nabla g_{i,\varepsilon_i} + \psi_i \Delta g_{i,\varepsilon_i}],$$

and since

$$\sum_i (\Delta\psi_i) g_i = \sum_i (\Delta\psi_i) g_\delta = 0,$$

we have

$$\Delta h = \sum_i [\Delta\psi_i (g_{i,\varepsilon_i} - g_i) + 2\sum_i \nabla\psi_i \cdot (\nabla g_{i,\varepsilon_i} - \nabla g_i) + \psi_i \Delta g_{i,\varepsilon_i}].$$

By (17), (18), we may choose ε_i to be sufficiently small so that

$$\|(-\Delta - \lambda)h\|_{L^1(D)} \le 2\|(-\Delta - \lambda)g_\delta\|_{L^1(D)}.$$

□

The above new criterion allows us to give a cleaner proof of Theorems 1.1 and 1.2. We will briefly sketch the proof here. First we start with the following volume comparison result.

Lemma 5.3. *Let $r(x)$ be the radial function to a fixed point p of M and suppose that the Laplacian of r satisfies* (5) *in the sense of distribution. Set $\delta > 0$. Then for any $0 < \varepsilon < 1/\delta$ there exists an $R_o > 0$ such that*

$$V(r+2\delta) - V(r) \leq \frac{2}{1-\delta\varepsilon}\,[V(r+\delta) - V(r)]$$

for any $r > R_o$.

Proof. We choose R_o such that for $r > R_o$ $m(r) \leq \varepsilon < 1/\delta$, where $m(r)$ is as in the proof of Lemma 4.3. Set t such that $r \leq t \leq r + \delta$. Stokes' Theorem implies that

$$\begin{aligned} \mathrm{vol}\,(\partial B(t+\delta)) - \mathrm{vol}\,(\partial B(t)) &= \int_{B(t+\delta)\setminus B(t)} \Delta r \\ &\leq \varepsilon\,[V(t+\delta) - V(t)] \leq \varepsilon\,[V(r+2\delta) - V(r)]. \end{aligned}$$

Integrating both sides of the inequality from r to $r + \delta$ we get

$$V(r+2\delta) - V(r+\delta) - [V(r+\delta) - V(r)] \leq \delta\varepsilon\,[V(r+2\delta) - V(r)]$$

which implies that

$$V(r+2\delta) - V(r+\delta) \leq \frac{1+\delta\varepsilon}{1-\delta\varepsilon}\,[V(r+\delta) - V(r)].$$

Adding $V(r+\delta) - V(r)$ to both sides of the inequality we get the desired result.

□

Alternative Proof of Theorem 1.1 and Theorem 1.2. Let $\tilde{\varepsilon} > 0$ be a small number and a, b be two large numbers such that $\sin(\sqrt{\lambda}a + \tilde{\varepsilon}) = \sin(\sqrt{\lambda}b + \tilde{\varepsilon}) = 0$. Define $\phi(x)$ to be the continuous function that vanishes on $\{r(x) \leq \sqrt{\lambda}a + \tilde{\varepsilon}\} \cup \{r(x) \geq \sqrt{\lambda}b + \tilde{\varepsilon}\}$ and is equal to $\sin(\sqrt{\lambda}\,r(x) + \tilde{\varepsilon})$ in between. The choice of $\tilde{\varepsilon} > 0$ is to ensure that $\phi \in \mathcal{C}_0^+(M)$, and in particular that property (3) of Definition 5.1 holds. Since for almost all $\tilde{\varepsilon}$, ϕ is in $\mathcal{C}_0^+(M)$, we assume, without loss of generality, that $\tilde{\varepsilon} = 0$ for the rest of the proof.

Let $D = \mathrm{supp}\,(\phi)$. Then on D, we have

$$(-\Delta - \lambda)\phi = -\sqrt{\lambda}\cos(\sqrt{\lambda}r)\Delta r$$

in the sense of distribution. Using Lemma 4.1 and our assumption on the Laplacian of r, we have

$$\|(-\Delta-\lambda)\phi\|_{L^1(D)} \leq \varepsilon \mathrm{vol}\,(D) + C\mathrm{vol}\,(\partial D)$$

for some constant C. Therefore

$$\|(-\Delta-\lambda)\phi\|_{L^1} \leq \|(-\Delta-\lambda)\phi\|_{L^1(D)} + \|\nabla\phi\|_{L^1(\partial D)} \leq \varepsilon \mathrm{vol}\,(D) + C\mathrm{vol}\,(\partial D)$$

for a possible larger constant C.

We would like to show that for a large enough there exists a constant C_o on the manifold, such that

$$C_o(V(b-\pi/\lambda) - V(a+\pi/\lambda)) \leq \|\phi\|_{L^2}$$

for some b big enough. Observing that the period of $\sin(\sqrt{\lambda}r)$ is $2\pi/\lambda$, we let N be the number such that $b = a + N\pi/\lambda$. Since $|\sin y| \geq 1/2$ for $y \in [k\pi+\pi/6, k\pi+5\pi/6]$ for any $k \in \mathbb{N}$, it follows that

$$\int_{B(b)\setminus B(a)} \sin^2(\sqrt{\lambda}r(x)) \geq \frac{1}{4}\sum_{i=0}^{N-1}[V(a+\frac{i\pi}{\sqrt{\lambda}}+\frac{5\pi}{6\sqrt{\lambda}}) - V(a+\frac{i\pi}{\sqrt{\lambda}}+\frac{\pi}{6\sqrt{\lambda}})].$$

Applying Lemma 5.3 to the right-hand-side of the above inequality with $\delta = \frac{4\pi}{6\sqrt{\lambda}}$ and $\varepsilon = \delta/2$ we get

$$\int_{B(b)\setminus B(a)} \sin^2(\sqrt{\lambda}r(x)) \geq \frac{1}{16}\sum_{i=0}^{N-1}[V(a+\frac{i\pi}{\sqrt{\lambda}}+\frac{9\pi}{6\sqrt{\lambda}}) - V(a+\frac{i\pi}{\sqrt{\lambda}}+\frac{\pi}{6\sqrt{\lambda}})]$$

for all $a > R_o(\lambda)$. It is easy to see that the right-hand-side is bounded below by

$$\frac{1}{16}[V(b+\frac{\pi}{2\sqrt{\lambda}}) - V(a+\frac{\pi}{6\sqrt{\lambda}})] \geq \frac{1}{16}[V(b-\frac{\pi}{\sqrt{\lambda}}) - V(a+\frac{\pi}{\sqrt{\lambda}})],$$

therefore the claim is true.

As in the proof of Lemma 5.1, our assumptions on the manifold imply that we can choose a and b appropriately such that $\varepsilon\mathrm{vol}\,(D) + C\mathrm{vol}\,(\partial D) \leq \tilde{C}\varepsilon(V(b-\pi/\lambda) - V(a+\pi/\lambda))$ for some $\tilde{C}$ independent of ε. Furthermore, the support of our test functions will be disjoint. Both theorems follow by the lower bound estimate on $\|\phi\|_{L^2}$. $\square$

6. Further Applications of the Generalized Weyl's Criterion

6.1. **Complete Shrinking Ricci Solitons.** A noncompact complete Riemannian manifold M with metric g is called a gradient shrinking Ricci soliton if there exists a smooth function f such that the Ricci tensor of the metric g is given by

$$R_{ij} + \nabla_i \nabla_j f = \rho\, g_{ij}$$

for some positive constant $\rho > 0$. By rescaling the metric we may rewrite the soliton equation as

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2}\, g_{ij}.$$

The scalar curvature R of a gradient shrinking Ricci soliton is nonnegative, and the volume growth of such manifolds (with respect to the Riemannian metric) is Euclidean. Hamilton [14] proved that the scalar curvature of a gradient shrinking Ricci soliton satisfies the equations

$$\nabla_i R = 2\, R_{ij}\, \nabla_j f,$$

$$R + |\nabla f|^2 - f = C_o$$

for some constant C_o. We may add a constant to f so that

$$R + |\nabla f|^2 - f = 0.$$

In [20], the authors proved that

(1) the L^1 essential spectrum of the Laplacian contains $[0, \infty)$;
(2) the L^2 essential spectrum of the Laplacian is $[0, \infty)$, if the scalar curvature has sub-quadratic growth.

Using our new Weyl Criterion, we removed the curvature condition:

Theorem 6.1. *The L^2 essential spectrum of a complete shrinking Ricci solution is $[0, \infty)$.*

Proof. It can be shown that $f(x) \geq 0$ and the key idea is to use $\rho(x) = 2\sqrt{f(x)}$ as an approximate distance function on the manifold, because of the special properties that it satisfies.

We define

$$D(r) = \{x \in M : \rho(x) < r\}$$

and set $V(r) = \mathrm{vol}\,(D(r))$. For some positive number y sufficiently large we consider the cut-off function $\chi : \mathbb{R}^+ \to \mathbb{R}$, smooth with support in $[0, y + 2]$ and such that

$\chi = 1$ on $[1, y+1]$ and $|\chi'|, |\chi''| \le C$. For any $\lambda > 0$ and large enough constants b, l we let

$$\phi(\rho) = \chi\left(\frac{\rho - b}{l}\right) e^{\sqrt{-1}\sqrt{\lambda}\rho}$$

which has support on $[b + l, b + l(y + 1)]$. Lu and Zhou [20, page 3289] demonstrate that for sufficiently large l and b

$$\int_M |\Delta\phi + \lambda\phi| \le \varepsilon V(b + (y + 2)l).$$

At the same time

$$\|\phi\|_{L^2}^2 \ge V(b + (y + 1)l) - V(b + l)$$

(note that the same holds true for the L^1 norm of ϕ). Arguing as is [20, Theorem 6] we conclude that there exists a y large enough such that

$$\int_M |\Delta\phi + \lambda\phi| \le 4\varepsilon\|\phi\|_{L^2}^2.$$

As in the previous section, we may also choose appropriate sequences of b_i, l_i such that the supports of the ψ_i are disjoint and condition *(2)* of Corollary 5.2 holds. Condition *(1)* is verified by the estimate above and the fact that $\|\phi_i\|_{L^\infty} = 1$. □

6.2. **Exhaustion functions on complete manifolds.** From what we have seen so far, it is apparent that two things are important when computing the essential spectrum of the Laplacian:

(1) The control of the L^1 norm of Δr;
(2) The control of the volume growth and decay of geodesic balls.

The same idea can be used for manifolds whose essential spectrum is not the half real line.

In the spirit of the results above, we are also able to modify a theorem of Elworthy and Wang [10]. We now consider manifolds on which there exists a continuous exhaustion function $\gamma \in \mathcal{C}(M)$ such that

(a) γ is unbounded above and is $\mathcal{C}^2$ smooth in the domain $\{\gamma > R\}$ for some $R > 0$ and

(b) $\mathrm{vol}\,(\{m_o < \gamma < n\}) < \infty$ for some m_o and any $n > m_o$ where the volume is measured with respect to the Riemannian metric.

For $t > 0$ and $c \in \mathbb{R}$ we define $B_t = \{\gamma(x) < t\}$ and set $dv_c = e^{-c\gamma}dv$. For $t \ge s$, let $U_c(s,t) = \mathrm{vol}_c(B_t \setminus B_s)$ where vol_c is the volume with respect to the measure dv_c.

We begin by stating the result of Elworthy and Wang for the sake of comparison.

Theorem 6.2 ([10, Theorem 1.1])**.** *Suppose that there exists a function $\gamma \in \mathcal{C}(M)$ that satisfies (a) and (b) and a constant $c \in \mathbb{R}$ such that*

$$\lim_{s\to\infty} \overline{\lim_{t\to\infty}}\, U_c(s,t)^{-1} \int_{B_t\setminus B_s} [(\Delta\gamma - c)^2 + (|\nabla\gamma|^2-1)^2]\, dv_c = 0 \tag{19}$$

and

$$\lim_{t\to\infty} \max\{U_c(m_o,t), U_c(t,\infty)^{-1}\}\, e^{-\varepsilon t} = 0 \qquad \text{for any } \varepsilon > 0. \tag{20}$$

Then $\sigma(-\Delta) \supset [c^2/4,\infty)$. When the above hold for $c=0$, then $\sigma(-\Delta) = [0,\infty)$.

Note that condition (20) implies that when $c=0$ the volume of the manifold grows and decays subexponentially, as was the case for us in the previous sections. The assumption here is that the weighted volume grows and decays subexponentially.

Our result is as follows:

Theorem 6.3. *Suppose that there exists a function $\gamma \in \mathcal{C}(M)$ that satisfies (a) and (b) and a constant $c \in \mathbb{R}$ such that*

$$\lim_{s\to\infty} \overline{\lim_{t\to\infty}}\; U_c(s,t)^{-1} \int_{B_t\setminus B_s} (|\Delta\gamma - c| + |\,|\nabla\gamma|^2-1|)\, dv_c = 0 \tag{21}$$

and

$$\lim_{t\to\infty} \max\{U_c(m_o,t), U_c(t,\infty)^{-1}\}\, e^{-\varepsilon t} = 0 \qquad \text{for any } \varepsilon > 0. \tag{22}$$

If (21) and (22) hold for $c=0$, then $\sigma(-\Delta) = [0,\infty)$.

In the case they hold for $c \neq 0$, we make the additional assumptions that the heat kernel of the Laplacian satisfies the pointwise bound

$$p_t(x,y) \le C t^{-m}\, e^{-\frac{(\gamma(x)-\gamma(y))^2}{4C_1 t} - \frac{d(x,y)^2}{4C_2 t} + \beta_1|\gamma(x)-\gamma(y)| + \beta_2 d(x,y) + \beta_3 t} \tag{23}$$

for some positive constants $m, C_1, C_2, \beta_1, \beta_2, \beta_3$, and that the Ricci curvature of the manifold is bounded below $\mathrm{Ric}(M) \ge -(n-1)K$ for a nonnegative number K. Then $\sigma(-\Delta) \supset [c^2/4,\infty)$.

In the case $c=0$, the main difference between our result and Theorem 1.1 of [10] is that we only need to control the L^1 norms of $|\Delta\gamma - c|$ and $||\nabla\gamma|^2-1|$ as in (21), instead of their L^2 norms (compare to (19)). Our assumption is weaker in various cases, for example when γ is the radial function where we know that its Laplacian

is not locally L^2 integrable when the manifold has a cut-locus, but it is locally L^1 integrable.

In the case $c \neq 0$, the additional assumption (23) is similar to requiring a uniform Gaussian bound for the heat kernel, but now with respect to the γ function as well. Such a bound is certainly true in the case of hyperbolic space with γ the radial function.

The proof uses similar estimates to those of Elworthy and Wang for the measures of annuli along the exhaustion function γ. We provide an outline of the argument with the necessary modifications.

Proof. Set $\lambda \geq c^2/4$ be a fixed number. For any $t > s$ we let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$, be a smooth cut-off function with support on $[s-1, t+1]$ and such that $\chi = 1$ on $[s,t]$ and $|\chi'|, |\chi''|$ bounded. Let $\lambda_c = \sqrt{\lambda - c^2/4}$ and define for $s \geq 0$

$$f(s) = e^{(i\lambda_c - c/2)\, s}.$$

Consider the test function

$$\phi_{s,t}(x) = \chi(\gamma(x))\, f(\gamma(x)).$$

We compute

$$\Delta\phi_{s,t} + \lambda\phi_{s,t} = (\chi'' f + 2\chi' f' + \chi f'')|\nabla\gamma|^2 + (\chi' f + \chi f')\Delta\gamma + \lambda\, \chi f.$$

Using the fact that $f'' + cf' + \lambda f = 0$ we obtain

$$\Delta\phi_{s,t} + \lambda\phi_{s,t} = (\chi'' f + 2\chi' f')|\nabla\gamma|^2 + (\chi' f)\Delta\gamma + \chi f'(\Delta\gamma - c|\nabla\gamma|^2) + \lambda\, \chi f(1 - |\nabla\gamma|^2).$$

Therefore there exists a constant C such that

$$|\Delta\phi_{s,t} + \lambda\phi_{s,t}| \leq Ce^{-c/2\gamma}\left[(|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)1_{\mathrm{spt}(B_{t+1}\setminus B_{s-1})} + 1_{\mathrm{spt}(\chi')}\right]. \tag{24}$$

For the rest of the estimates, we will repeatedly use

$$\lim_{s,t\to\infty} (U_c(s-1,s) + U_c(t,t+1))/U_c(s,t) = 0, \tag{25}$$

which follows from (22).

Using (24), we have

$$\begin{aligned} |(\phi_{s,t}, \Delta\phi_{s,t} + \lambda\phi_{s,t})| \leq & C\int_{B_{t+1}\setminus B_{s-1}} (|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)\, dv_c \\ & + C(U_c(s-1,s) + U_c(t,t+1)). \end{aligned} \tag{26}$$

We observe that

$$\begin{aligned}\frac{1}{U_c(s,t)}\int_{B_{t+1}\setminus B_{s-1}} &(|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)\,dv_c \\ &= \Big[1 + \frac{U_c(s-1,s)+U_c(t,t+1)}{U_c(s,t)}\Big] \\ &\quad\cdot \frac{1}{U_c(s-1,t+1)}\int_{B_{t+1}\setminus B_{s-1}} (|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)\,dv_c,\end{aligned}$$

which tends to zero as $s, t \to \infty$ by (25) and assumption (21). Since $\|\phi_{s,t}\|^2_{L^2} \geq U_c(s,t)$, inequality (26), the above estimate and (25) imply that

$$\lim_{s,t\to\infty} |(\phi_{s,t}, \Delta\phi_{s,t} + \lambda\phi_{s,t})|/\|\phi_{s,t}\|^2_{L^2} = 0. \tag{27}$$

When $c = 0$, we choose appropriate sequences of $s_n, t_n \to \infty$ such that condition *(2)* of Corollary 5.2 holds. Condition *(1)* of the Corollary follows from (27) and the fact that the functions ϕ_{s_n,t_n} are bounded. Therefore, $\lambda_0 = \sqrt{\lambda}$ belongs to the essential L^2 spectrum. Given that λ is any nonnegative number, the result follows.

In the case $c \neq 0$, we will apply Corollary 2.2. For a fixed natural number $i > m$ and $\alpha > 0$ we have that the integral kernel of $(-\Delta + \alpha)^{-i}$, $g^i_\alpha(x,y)$, is given by

$$g^i_\alpha(x,y) = C(n)\int_0^\infty p_t(x,y)\, t^{i-1}\, e^{-\alpha t}\, dt.$$

On the other hand, it is a property of the exponential function that for any $\beta_4, \beta_5 \in \mathbb{R}$

$$e^{-\frac{(\gamma(x)-\gamma(y))^2}{4C_1 t}} \leq e^{-\beta_4|\gamma(x)-\gamma(y)|} e^{C_1\,\beta_4^2 t}$$

and

$$e^{-\frac{d(x,y)^2}{4C_2 t}} \leq e^{-\beta_5 d(x,y)} e^{C_2\,\beta_5^2 t}.$$

Combining the above, we have that for any $N > m$ and $\beta_4, \beta_5 > 0$ there exists an $\alpha > 0$ large enough, and a constant C such that

$$g^i_\alpha(x,y) \leq C\, e^{-\beta_4\,|\gamma(x)-\gamma(y)| - \beta_5 d(x,y)}$$

for $i = N, N+1$. As a result, for any $t > s > 2$

$$\begin{aligned}\int_{B_{t+1}\setminus B_{s-1}} g^i_\alpha(x,y) e^{-c/2\gamma(y)} dy &\leq C\int_{B_{t+1}\setminus B_{s-1}} e^{-\beta_4|\gamma(x)-\gamma(y)| - \beta_5 d(x,y)}\, e^{-c/2\gamma(y)} dy \\ &\leq C\, e^{-c/2\gamma(x)}\end{aligned}$$

after choosing $\beta_4 = |c|/2$ and $\beta_5 > \sqrt{K}$. This estimate together with (25) also give

$$\begin{aligned}|(\,(-\Delta+\alpha)^{-i}\phi_{s,t}, \Delta\phi_{s,t} + \lambda\phi_{s,t})| \leq & C\int_{B_{t+1}\setminus B_{s-1}} (|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)\, dv_c \\ & + C(U_c(s-1,s) + U_c(t,t+1)).\end{aligned}$$

As a result,

$$\lim_{s,t\to\infty} |(\,(-\Delta+\alpha)^{-i}\phi_{s,t}, \Delta\phi_{s,t} + \lambda\phi_{s,t})|/\|\phi_{s,t}\|_{L^2}^2 = 0. \tag{28}$$

Choosing appropriate sequences of $s_n, t_n \to \infty$ and setting $\psi_n = \phi_{s_n,t_n}/\|\phi_{s_n,t_n}\|_{L^2}$, conditions *(1)* and *(4)* of Corollary 2.2 hold for the functions ψ_n. That *(2)* and *(3)* also hold follows from (27) and (28) respectively.

□

7. Some Remarks on the Theorem of Sturm

We end our article with some final remarks on the relation of the spectrum of the Laplacian to the volume growth of the manifold.

Definition 7.1. *The volume of a manifold M grows uniformly subexponentially, if and only if for any $\varepsilon > 0$ there exists a finite constant $C(\varepsilon)$ such that, for all $r > 0$ and $x \in M$*

$$\mathrm{vol}\,(B_x(r)) \leq C(\varepsilon)\,\mathrm{vol}\,(B_x(1))\,e^{\varepsilon r}. \tag{29}$$

Sturm proves the following Proposition

Proposition 7.1 (Sturm, [21, Proposition 1])**.** *If M has uniformly subexponential volume growth, then for every $\varepsilon > 0$*

$$\sup_{x\in M}\int_M e^{-\varepsilon d(x,y)}[\mathrm{vol}\,(B_x(1))]^{-1/2}\cdot[\mathrm{vol}\,(B_y(1))]^{-1/2}dy < \infty. \tag{30}$$

Sturm's goal was to find sufficient conditions on a manifold such that the spectrum of the Laplacian on L^q is independent of $q \in [1,\infty]$. He proves that such sufficient conditions are that Ricci curvature is bounded below and that (30) holds for all $\varepsilon > 0$. We will prove in fact that on a manifold with Ricci curvature bounded below, (30) holds for all $\varepsilon > 0$ if and only if the volume of the manifold grows uniformly subexponetially. In other words, Sturm's L^q independence result has only been proved for manifolds whose volume grows uniformly subexponetially.

Proposition 7.2. *Let M be a complete noncompact Riemannian manifold whose Ricci curvature has a lower bound. Then the following four conditions are equivalent*[1]

(a) *The volume of M grows uniformly subexponentially.*

(b) *For every $\varepsilon > 0$,*

$$\sup_{x\in M} \int_M e^{-\varepsilon d(x,y)} [\mathrm{vol}\,(B_x(1))]^{-1} \cdot dy < \infty.$$

(c) *For every $\varepsilon > 0$,*

$$\sup_{x\in M} \int_M e^{-\varepsilon d(x,y)} [\mathrm{vol}\,(B_y(1))]^{-1} \cdot dy < \infty.$$

(d) *For every $\varepsilon > 0$,*

$$\sup_{x\in M} \int_M e^{-\varepsilon d(x,y)} [\mathrm{vol}\,(B_x(1))]^{-1/2} \cdot [\mathrm{vol}\,(B_y(1))]^{-1/2} dy < \infty.$$

Proof. We first observed that, if (a) is valid, then from (29), we have

$$\mathrm{vol}\,(B_y(1)) \le C(\varepsilon) e^{\varepsilon(d(x,y)+1)} \mathrm{vol}\,(B_x(1))$$

for any $x, y \in M$. Since $\varepsilon > 0$ is arbitrary, $(b), (c), (d)$ are equivalent if (a) is valid.

Now we assume (a), then we have

$$\int_M e^{-\varepsilon d(x,y)} dy \le \sum_{i=0}^{\infty} e^{-\varepsilon i}\, \mathrm{vol}\,(B_x(i+1) \backslash B_x(i))$$
$$\le \sum_{i=0}^{\infty} e^{-\varepsilon i} C(\varepsilon/2) e^{\varepsilon/2(i+1)} \mathrm{vol}\,(B_x(1)) \le C\, \mathrm{vol}\,(B_x(1)),$$

where C is a constant depending only on ε. This proves that (a) implies $(b), (c), (d)$.

We will now prove that $(b), (c), (d)$ are in fact equivalent whenever the Ricci curvature is bounded below. We begin by assuming that (d) is valid. Then for fixed points $x, z \in M$, we have

$$\int_{B_z(1)} e^{-\varepsilon d(x,y)} [\mathrm{vol}\,(B_y(1))]^{-1/2} dy \le C [\mathrm{vol}\,(B_x(1))]^{1/2}.$$

[1] That (a) implies (d) was pointed out in [21, page 444]. We include the proof for the sake of completeness.

Since $d(y,z) < 1$, we have $B_y(1) \subset B_z(2)$. Therefore from the above inequality, we have

$$\mathrm{vol}\,(B_z(2))^{-1/2} \cdot \int_{B_z(1)} e^{-\varepsilon d(x,y)} dy \le C[\mathrm{vol}\,(B_x(1))]^{1/2},$$

and hence

$$\mathrm{vol}\,(B_z(2))^{-1/2} \cdot e^{-\varepsilon(d(x,z)+1)}\,\mathrm{vol}\,(B_z(1)) \le C[\mathrm{vol}\,(B_x(1))]^{1/2}.$$

Since the Ricci curvature has a lower bound, by the volume comparison theorem, we have

$$\mathrm{vol}\,(B_z(2)) \le C\mathrm{vol}\,(B_z(1))$$

for a constant C depending only on the dimension and the lower bound of the Ricci curvature. Combining the above inequalities, we obtain

$$\mathrm{vol}\,(B_z(1)) \le Ce^{2\varepsilon d(x,z)}\mathrm{vol}\,(B_x(1))$$

for any x, z. Similarly, if either (b) or (c) is valid, then the above inequality is true. Thus $(b), (c), (d)$ are equivalent under the condition that the Ricci curvature has a lower bound.

Finally, we prove that (b) implies (a). This follows from

$$\mathrm{vol}\,(B_x(r)) \le e^{\varepsilon r} \int_M e^{-\varepsilon d(x,y)}\, dy \le C(\varepsilon) e^{\varepsilon r}\mathrm{vol}\,(B_x(1)).$$

□

The L^q spectrum of the Laplacian is known however to depend on q on certain manifolds with exponential volume growth. Davies, Simon and Taylor prove such dependence in the case of the hyperbolic space and certain Kleinian groups [6]. We state their result in the former case for simplicity:

Theorem 7.1 ([6])**.** *Consider the Laplacian Δ on functions over the hyperbolic space $\mathbb{H}^{N+1}$. Then for $q \in [1,\infty)$ the L^q essential spectrum of Δ is exactly the set of points to the right of the parabola*

$$Q_p = \{-(\frac{N}{q} + is)(\frac{N}{q} + is - N) \mid s \in \mathbb{R}\}.$$

For $q = \infty$, the L^∞ spectrum is defined such that it coincides with the L^1 spectrum.

The important observation here, is that the L^q spectrum varies among the non-dual q. Based on this, we make the following conjecture.

Conjecture 1. *Let M be a complete noncompact Riemannian manifold with Ricci curvature bounded below. Then the L^q spectra are the same if and only if the volume of the manifold grows uniformly subexponentially.*

The only if part is Sturm's Theorem. We believe that the if part is related to the wave kernel, and we shall investigate this in a forthcoming paper.

References

[1] N. Charalambous and Z. Lu, *The essential spectrum of the Laplacian.* in preparation.

[2] N. Charalambous, *On the equivalence of heat kernel estimates and logarithmic Sobolev inequalities for the Hodge Laplacian*, J. Differential Equations **233** (2007), no. 1, 291–312.

[3] J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982), no. 1, 15–53. MR658471 (84b:58109)

[4] Z. H. Chen and Z. Q. Lu, *Essential spectrum of complete Riemannian manifolds*, Sci. China Ser. A **35** (1992), no. 3, 276–282. MR1183713 (93k:58221)

[5] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989.

[6] E. B. Davies, B. Simon, and M. Taylor, *L^p spectral theory of Kleinian groups*, J. Funct. Anal. **78** (1988), no. 1, 116–136.

[7] Y. Dermenjian, M. Durand, and V. Iftimie, *Spectral analysis of an acoustic multistratified perturbed cylinder*, Comm. Partial Differential Equations **23** (1998), no. 1-2, 141–169. MR1608508 (99a:35035)

[8] H. Donnelly, *Spectrum of the Laplacian on asymptotically Euclidean spaces*, Michigan Math. J. **46** (1999), no. 1, 101–111.

[9] ———, *Exhaustion functions and the spectrum of Riemannian manifolds*, Indiana Univ. Math. J. **46** (1997), no. 2, 505–527, DOI 10.1512/iumj.1997.46.1338. MR1481601 (99b:58230)

[10] K. D. Elworthy and F.-Y. Wang, *Essential spectrum on Riemannian manifolds*, Recent developments in stochastic analysis and related topics, World Sci. Publ., Hackensack, NJ, 2004, pp. 151–165.

[11] J. F. Escobar, *On the spectrum of the Laplacian on complete Riemannian manifolds*, Comm. Partial Differential Equations **11** (1986), no. 1, 63–85.

[12] J. F. Escobar and A. Freire, *The spectrum of the Laplacian of manifolds of positive curvature*, Duke Math. J. **65** (1992), no. 1, 1–21.

[13] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[14] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995, pp. 7–136.

[15] J. Y. Li, *Spectrum of the Laplacian on a complete Riemannian manifold with nonnegative Ricci curvature which possess a pole*, J. Math. Soc. Japan **46** (1994), no. 2, 213–216, DOI 10.2969/jmsj/04620213. MR1264938 (95g:58248)

[16] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu; Translated from the Chinese by Ding and S. Y. Cheng; Preface translated from the Chinese by Kaising Tso.

[17] P. Li and J. Wang, *Complete manifolds with positive spectrum*, J. Differential Geom. **58** (2001), no. 3, 501–534.

[18] D. Krejčiřík, *Twisting versus bending in quantum waveguides*, Analysis on graphs and its applications, Proc. Sympos. Pure Math., vol. 77, Amer. Math. Soc., Providence, RI, 2008, pp. 617–637. See arXiv:0712.3371v2 [math-ph] (2009) for a corrected version. MR2459893 (2010g:81112)

[19] D. Krejčiřík and J. Kříž, *On the spectrum of curved quantum waveguides*, Publ. RIMS, Kyoto University **41** (2005), 757–791.

[20] Z. Lu and D. Zhou, *On the essential spectrum of complete non-compact manifolds*, J. Funct. Anal. **260** (2011), no. 11, 3283–3298, DOI 10.1016/j.jfa.2010.10.010. MR2776570 (2012e:58058)

[21] K.-T. Sturm, *On the L^p-spectrum of uniformly elliptic operators on Riemannian manifolds*, J. Funct. Anal. **118** (1993), no. 2, 442–453, DOI 10.1006/jfan.1993.1150. MR1250269 (94m:58227)

[22] J. Wang, *The spectrum of the Laplacian on a manifold of nonnegative Ricci curvature*, Math. Res. Lett. **4** (1997), no. 4, 473–479.

[23] D. T. Zhou, *Essential spectrum of the Laplacian on manifolds of nonnegative curvature*, Internat. Math. Res. Notices **5** (1994), 209 ff., approx. 6 pp. (electronic), DOI 10.1155/S1073792894000231. MR1270134 (95g:58250)

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